Stabilizers of group actions (D+F 2.2)

Notice that the centralizer and normalizer are subgroups determined by G acting on subsets of itself by "conjugation" (we'll come back to this shortly).

We can generalize these subgroups to arbitrary group actions.

Let G be a group acting on a set A.

Def: Let $a \in A$. The stabilizer of a in G is the set

$$G_a = \{ g \in G \mid g \cdot a = a \}$$

Claim: $G_a \leq G$.

Pf: $1 \cdot a = a$, so $G_a \neq \emptyset$. If $x, y \in G_s$, then

$x^{-1} \cdot a = x^{-1}(x \cdot a) = 1 \cdot a = a$, so $x^{-1} \in G_s$.

$(xy) \cdot a = a$. so $G_a \leq G$. $\square$

Ex: Let $G = S_4$. Consider the action of G on $\{1,2,3,4\}$ given by $\sigma \cdot a = \sigma(a)$. What is G_4?

G_4 consists of cycle decompositions that don't contain 4.

i.e. $G_4 = \{1, (12), (13), (23), (123), (132)\}$. Note that this is naturally isomorphic to S_3.

Def: The <u>kernel</u> of a group action is $\{g \in G \mid g \cdot a = a \ \forall a \in A\}$

Note that this is the set of all elements of g that act as the identity on A.

i.e. if $\sigma_g : A \to A$ is defined (as before) as $\sigma_g(a) = g \cdot a$,

then the kernel is the set of g s.t. σ_g = identity.

In other words, the kernel of the group action is equal to the kernel of the map we defined

$$G \to S_A \text{ by } g \mapsto \sigma_g.$$

So the kernel of the group action is equal to the kernel of a group homomorphism, so it must be a subgroup.

Just as before, the action is faithful $\iff$ its kernel $= 1$.

Conjugation as a group action

Let G be a group. Let $S = \mathcal{P}(G)$ = the set of all subsets of G.

Let G act on S by <u>conjugation</u>. That is, if $A \in S$,

define

$$g \cdot A = gAg^{-1} = \{h \in G \mid h = gag^{-1} \text{ for some } a \in A\} \in S.$$

Claim: This is in fact a group action.

Pf: If $A \in S$, then $1 \cdot A = A$.

For $g, h \in G$, we have

$$(gh) \cdot A = (gh)A(gh)^{-1} = (gh)A(h^{-1}g^{-1}) = \{k \mid k = ghah^{-1}g^{-1}, \text{ some } a \in A\}$$
$$= g(hAh^{-1})g^{-1} = g \cdot (hAh^{-1}) = g \cdot (h \cdot A). \quad \square$$

Now, for any subset $A \subseteq G$, we have $A \in \mathcal{P}(G)$.
So $G_A = \{g \mid gAg^{-1} = A\} = N_G(A)$. That is,

The normalizer of A is equal to the stabilizer of A under the action of conjugation.

We can also act on individual elements by conjugation:

Claim: G acts on itself by $g \cdot a = gag^{-1}$.

Pf: $1 \cdot a = a$, and if $g, h \in G$,

$$(gh) \cdot a = (gh)a(gh)^{-1} = ghah^{-1}g^{-1} = g(h \cdot a)g^{-1} = g \cdot (h \cdot a). \quad \square$$

Note that the kernel of this action is exactly

$$\{g \in G \mid gag^{-1} = a \ \forall a \in G\} = Z(G).$$

Moreover, for any subset $A \subseteq G$, $N_G(A)$ acts on A by conjugation (by construction).

Then the kernel of this action is

$$\{g \in N_G(A) \mid gag^{-1} = a \quad \forall a \in A\} = N_G(A) \cap C_G(A)$$

But $C_G(A) \leq N_G(A)$, so

The kernel of the action of $N_G(A)$ on A by conjugation is equal to $C_G(A)$.